PROOF THAT $g(\ln\tau)$ FOR SINGLE RELAXATION TIME EXPRESSIONS FOR $Q^* = Q' - Q''$ ARE DIRAC DELTA FUNCTIONS (8/24/24)

It needs to be shown that the integral of $g(\ln\tau)$ is unity. All integrals are taken from https://www.wolframalpha.com/calculators/integral-calculator/

(1) From $Q'' = \omega\tau/(1+\omega^2\tau^2)$ and $g(\ln\tau) = \mathrm{Re}\left\{Q''(\tau/\tau_0)\exp(\pm i\pi/2)\right\}$, let $\theta = \tau/\tau_0$ and $\exp(i\pi/2) \to \lim_{\varepsilon\to 0}(i+\varepsilon)$, so that

$$g(\ln\tau) = \lim_{\varepsilon\to 0}\mathrm{Re}\left\{\left[\frac{\theta(i+\varepsilon)}{1+\theta^2(i+\varepsilon)^2}\right]\right\} = \lim_{\varepsilon\to 0}\mathrm{Re}\left[\frac{\theta(i+\varepsilon)}{1-\theta^2+2i\varepsilon\theta^2}\right]$$

$$= \lim_{\varepsilon\to 0}\mathrm{Re}\left\{\left[\frac{\theta(i+\varepsilon)\left[1-\theta^2-2i\varepsilon\theta^2\right]}{\left(1-\theta^2\right)^2}\right]\right\} = \lim_{\varepsilon\to 0}\left[\frac{\varepsilon\left(\theta-\theta^3\right)+2\varepsilon\theta^3}{\left(1-\theta^2\right)^2}\right]$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\varepsilon\left(\theta+\theta^3\right)}{\left(1-\theta^2\right)^2}\right].$$

The two indefinite integrals are

$$I_1 = \varepsilon\int\frac{\theta d\theta}{\left(1-\theta^2\right)^2} = \frac{\varepsilon}{2\left(1-\theta^2\right)}$$

and

$$I_2 = \varepsilon\int\frac{\theta^3 d\theta}{\left(1-\theta^2\right)^2} = \frac{1}{2}\left[\frac{\varepsilon}{\left(1-\theta^2\right)} + \varepsilon\ln\left(1-\theta^2\right)\right].$$

Both these integrals must be evaluated through the singularity at $\theta = 1$ as the Cauchy principal values. The first integral is

$$I_1 = \varepsilon\lim_{\delta\to 0}\left\{\frac{1}{2\left(\theta^2-1\right)^2}\Bigg|_{1-\delta}^{1+\delta}\right\} = \varepsilon\lim_{\delta\to 0}\left\{\frac{1}{2\left[(1+\delta)^2-1\right]} - \frac{1}{2\left[(1-\delta)^2-1\right]}\right\}$$

$$= \varepsilon\lim_{\delta\to 0}\left[\frac{1}{2}\left(\frac{1}{2\delta}-\frac{1}{-2\delta}\right)\right] = \varepsilon\lim_{\delta\to 0}\left(\frac{1}{2\delta}\right).$$

The first part of the second integral is the same as the first integral and the second part is

$\varepsilon \ln\left(\theta^2 - 1\right)\Big|_{1-\delta}^{1+\delta} = \varepsilon \ln(-1) = \varepsilon i\pi$ s .The two integrals therefore add up to $\varepsilon\left[\lim_{\delta \to 0}\left(\frac{1}{\delta} + i\pi\right)\right]$, so that

$$\int g(\ln \tau) = \lim_{\varepsilon \to 0}\left\{\varepsilon\left[\lim_{\delta \to 0}\left(\frac{1}{\delta} + i\pi\right)\right]\right\} = \left(\frac{\varepsilon}{\delta}\right).$$

Only if $\delta = \varepsilon$ is the area unity. This constraint is physically necessary and might be mathematically rigorous (still being researched).